

# On the distance from a normal matrix polynomial to the set of matrix polynomials with a prescribed multiple eigenvalue

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## Abstract

This paper concerns the bounds for spectral norm distance from a normal matrix polynomial  $P(\lambda)$  to the set of matrix polynomials that have  $\mu$  as a multiple eigenvalue. Also construction of associated perturbations of  $P(\lambda)$  is considered.

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## 1 Introduction

For a complex  $n \times n$  matrix  $A$  and a given complex number  $\mu$ , the spectral norm distance from  $A$  to the matrices that have  $\mu$  as a multiple eigenvalue was proved by A. M. Malyshev [10]. Malyshev's results extended by Lippert [8] and Gracia [4] and they compute a 2-norm distance from  $A$  to the set of matrices with two prescribed eigenvalue. In 2004, Ikramov and Nazari [5] show that Malyshev's formula not a viable method for the case of normal matrices. Also an operative manner for this case was introduced by them. Moreover, Nazari and Rajabi [9] noted this issue for distance from a normal matrix  $A$  to the set of matrices with two prescribed eigenvalues.

In 2008, a spectral norm distance from a matrix polynomial  $P(\lambda)$  to the matrix polynomials that have  $\mu$  as an eigenvalue of geometric multiplicity at least  $\kappa$ , and a

distance from  $P(\lambda)$  to the matrix polynomials that have  $\mu$  as a multiple eigenvalue was introduced by Papathanasiou and Psarrakos [12]. They computed the first distance and also obtained bounds for the second one, constructing an associated perturbations of  $P(\lambda)$ . In this paper, at first we illustrate by an example that procedure described in [12] not a efficient method for the case of normal matrix polynomials. Then a suitable process is presented.

## 2 Preliminaries

In this section some definitions of matrix polynomials are reviewed. A good reference for the theory of matrix polynomials is [2]. Moreover, we illustrate by a specific example that method described in [12] is useless for normal matrix polynomials.

**Definition 2.1.** For  $A_j \in \mathbb{C}^{n \times n} (j = 0, 1, \dots, m)$  with  $\det(A_m) \neq 0$  and a complex variable  $\lambda$ , we define the matrix polynomial  $P(\lambda)$  as

$$P(\lambda) = A_m \lambda^m + A_{m-1} \lambda^{m-1} + \dots + A_1 \lambda + A_0. \quad (1)$$

If for a scalar  $\mu$  and some nonzero vector  $v \in \mathbb{C}^n$ , it holds that  $P(\mu)v = 0$ , then the scalar  $\mu$  called an *eigenvalue* of  $P(\lambda)$  and the vector  $v$  is known as a *right eigenvector* of  $P(\lambda)$  corresponding to  $\mu$ . Similarly, nonzero vector  $\nu \in \mathbb{C}^n$  is known as a left eigenvector of  $P(\lambda)$  corresponding to  $\mu$  if we have  $\nu^* P(\mu) = 0$ . Multiplicity of  $\mu$  as a root of the scalar polynomial  $\det P(\lambda)$  called as its *algebraic multiplicity* and number of linear independent eigenvectors corresponding to  $\mu$  is *geometric multiplicity*. algebraic multiplicity of an eigenvalue is always greater or equal to its geometric multiplicity. If algebraic and geometric multiplicities of an eigenvalue are equal, this eigenvalue called *semisimple*, otherwise it named *defective*.

**Definition 2.2.** Let  $P(\lambda)$  be a matrix polynomial as in (1), if there exists a unitary matrix  $U \in \mathbb{C}^{n \times n}$  such that  $U^* P(\lambda) U$  is diagonal for all  $\lambda \in \mathbb{C}$ , then  $P(\lambda)$  named *weakly normal*. Moreover,  $P(\lambda)$  is called *normal* if all the eigenvalues of  $P(\lambda)$  are semisimple.

For the matrix polynomials of the form  $A + \lambda B$ , two concepts weakly normal and normal are coincide. The matrix polynomial  $P(\lambda)$  is weakly normal if and only if for every  $\mu \in \mathbb{C}$  the matrix  $P(\mu)$  is a normal matrix. Moreover, all of the coefficient matrices  $A_i, (i = 1, \dots, m)$  in (1) also all linear combinations of them are normal matrices [11].

Here, some of results obtained in sections 4 and 5 of [12] are reviewed briefly.

For the matrix polynomial  $P(\lambda)$  in (1) and a complex variable  $\gamma$ , Papathanasiou and Psarrakos [12] define

$$F[P(\lambda); \gamma] = \begin{bmatrix} P(\lambda) & 0 \\ \gamma P'(\lambda) & P(\lambda) \end{bmatrix}_{2n \times 2n},$$

where  $P'(\lambda)$  denotes the derivative of  $P(\lambda)$  with respect to  $\lambda$ .

**Lemma 2.3.** [12]. *Let  $\mu \in \mathbb{C}$  and  $\gamma_* > 0$  be a point where the singular value  $s_{2n-1}(F[P(\mu); \gamma])$  attains its maximum value, and  $s_* = s_{2n-1}(F[P(\mu); \gamma_*]) > 0$ . Then there exist a pair  $\begin{bmatrix} u_1(\gamma_*) \\ u_2(\gamma_*) \end{bmatrix}, \begin{bmatrix} v_1(\gamma_*) \\ v_2(\gamma_*) \end{bmatrix} \in \mathbb{C}^{2n}$  ( $u_k(\gamma_*), v_k(\gamma_*) \in \mathbb{C}^n$ ,  $k = 1, 2$ ) of left and right singular vectors of  $s_*$  respectively, such that*

1.  $u_2^*(\gamma_*)P'(\mu)v_1(\gamma_*) = 0$ , and
2. the  $n \times 2$  matrices  $U(\gamma_*) = [u_1(\gamma_*) \ u_2(\gamma_*)]_{n \times 2}$  and  $V(\gamma_*) = [v_1(\gamma_*) \ v_2(\gamma_*)]_{n \times 2}$  satisfy  $U^*(\gamma_*)U(\gamma_*) = V^*(\gamma_*)V(\gamma_*)$ .

Suppose that weights  $w = \{\omega_0, \omega_1, \dots, \omega_m\}$  are given, such that  $w$  is a set of nonnegative coefficients with  $\omega_0 > 0$ . The scalar polynomial  $w(\lambda)$  corresponding to the weights is defined in the form

$$w(\lambda) = w_m \lambda^m + \dots + w_1 \lambda + w_0.$$

Moreover, consider the matrix

$$\Delta_{\gamma_*} = -s_* U(\gamma_*) \begin{bmatrix} 1 & -\gamma_* \phi \\ 0 & 1 \end{bmatrix} V(\gamma_*)^\dagger,$$

where  $V(\gamma_*)^\dagger$  is the *Moore-Penrose pseudoinverse* of  $V(\gamma_*)$  and the quantity  $\phi$  is

$$\phi = \frac{w'(|\mu|)}{w(|\mu|)} \frac{\bar{\mu}}{|\mu|},$$

if  $\mu = 0$ , then by convention we set  $\frac{\bar{\mu}}{|\mu|} = 0$ .

If  $\gamma_* > 0$  and  $\mu \in \mathbb{C}$  is not an eigenvalue of  $P'(\lambda)$ , then [12, Theorem 19] implies that matrix polynomial  $Q_{\gamma_*}(\lambda) = P(\lambda) + \Delta_{\gamma_*}(\lambda)$  has  $\mu$  as a defective eigenvalue. Where

$$\Delta_{\gamma_*}(\lambda) = \sum_{j=0}^m \left( \frac{w_j}{w(|\mu|)} \left( \frac{\bar{\mu}}{|\mu|} \right)^j \Delta_{\gamma_*} \right) \lambda^j.$$

Now for a specific example, let us consider the normal matrix polynomial  $P(\lambda)$  as mentioned in [11, section 3] of the form

$$P(\lambda) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lambda^2 + \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \lambda + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \quad (2)$$

Consider the set of weights  $w = \{1, 1, 1\}$  and  $\mu = 3$ . By applying the procedure described sections 4 and 5 of [12] (and reviewed briefly above), it can be obtained that  $s_5(F[P(3); \gamma])$  attains its maximum value at  $\gamma_* = 1$ , and  $s_* = s_5(F[P(3); \gamma_*]) = 4$ . Also we have find the  $Q(\gamma_*)$  as following

$$Q_{\gamma_*}(\lambda) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.6013 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lambda^2 + \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1.3987 & 0 \\ 0 & 0 & 3 \end{bmatrix} \lambda + \begin{bmatrix} 2 & 0 & 0 \\ 0 & -0.3987 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

it is straightforward to see that  $\mu = 3$  is not an eigenvalue of  $Q_{\gamma_*}(\lambda)$ . Furthermore,

$$u_2^*(\gamma_*)P'(\mu)v_1(\gamma_*) = -1.5385, \quad \text{and} \quad \|U^*(\gamma_*)U(\gamma_*) - V^*(\gamma_*)V(\gamma_*)\|_2 = 0.3846,$$

imply that none of propositions of the Lemma 2.3 is not confirmed.

### 3 Normal matrix polynomial

As was mentioned in the previous section, described method in [12] is not efficient for the case of normal matrix polynomials. Moreover, both results of Lemma 2.3 are violated. In fact violation of the first part of Lemma 2.3 is cause of violation of the second part. By similar analysis fulfilled in [6] it can be showed that when  $\gamma \geq 0$  started to rise  $s_{2n-1}(F[P(\mu); \gamma])$  increases and  $s_{2n-2}(F[P(\mu); \gamma])$  decreases, and where to next this process will reversed. Consequently, there exists a point such as  $\gamma_*$  where  $s_{2n-1}(F[P(\mu); \gamma])$  attains its maximum value. Furthermore, at  $\gamma = \gamma_*$  we have  $s_* = s_{2n-1}(F[P(\mu); \gamma_*]) = s_{2n-2}(F[P(\mu); \gamma_*])$ , that means  $s_*$  is a multiple singular value of  $F[P(\mu); \gamma]$ . The graphs of the  $s_5(F[P(3); \gamma])$  and  $s_4(F[P(3); \gamma])$  for  $\gamma \in [0, 10]$  are plotted in Fig 1.

Suppose that  $u^{2n-1}(\gamma_*)$ ,  $v^{2n-1}(\gamma_*)$  and  $u^{2n-2}(\gamma_*)$ ,  $v^{2n-2}(\gamma_*)$  are a pair of singular vectors of  $s_{2n-1}(F[P(\mu); \gamma_*])$  and  $s_{2n-2}(F[P(\mu); \gamma_*])$ , respectively. As we have seen, computation of first proposition in the Lemma 2.3 for  $s_{2n-1}(F[P(\mu); \gamma_*])$  yields

$$u_2^{(2n-1)}(\gamma_*)^* P'(\mu) v_1^{(2n-1)}(\gamma_*) = -1.5385, \quad (3)$$

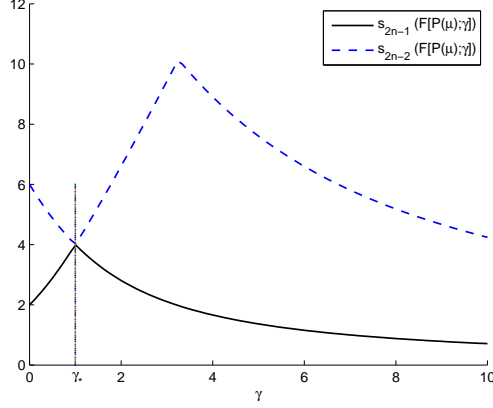


Fig 1:  $s_{2n-1}(F[P(\mu); \gamma])$  and  $s_{2n-2}(F[P(\mu); \gamma])$

also similar computations but for  $s_{2n-2}(F[P(\mu); \gamma_*])$  leads to

$$u_2^{(2n-2)}(\gamma_*)^* P'(\mu) v_1^{(2n-2)}(\gamma_*) = 2.4, \quad (4)$$

According to Lemma 16 of [12], it is straightforward to see that  $u_2(\gamma)^* P'(\mu) v_1(\gamma)$  can be explained as the derivative of the corresponding singular value with respect to  $\gamma$ . Therefore, negativity and positivity of the numbers in (3) and (4) (respectively) means that  $s_{2n-1}(F[P(\mu); \gamma])$  and  $s_{2n-2}(F[P(\mu); \gamma])$  are decreasing and increasing functions, respectively.

Hereafter, we are looking for a pair of vectors  $u$  and  $v$  in the form

$$u(\gamma_*) = \alpha u^{(2n-1)}(\gamma_*) + \beta u^{(2n-2)}(\gamma_*), \quad \text{and} \quad v(\gamma_*) = \alpha v^{(2n-1)}(\gamma_*) + \beta v^{(2n-2)}(\gamma_*), \quad (5)$$

such as satisfy

$$u_2(\gamma_*)^* P'(\mu) v_1(\gamma_*) = 0. \quad (6)$$

Where for the two scalars  $\alpha$  and  $\beta$  we have  $|\alpha|^2 + |\beta|^2 = 1$ . For doing this, substituting  $u$  and  $v$  in (5) into (6) leads to

$$\begin{bmatrix} \bar{\alpha} & \bar{\beta} \end{bmatrix} M \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0, \quad (7)$$

where

$$M = \begin{bmatrix} u_2^{(2n-1)}(\gamma_*)^* P'(\mu) v_1^{(2n-1)}(\gamma_*) & u_2^{(2n-1)}(\gamma_*)^* P'(\mu) v_1^{(2n-2)}(\gamma_*) \\ u_2^{(2n-2)}(\gamma_*)^* P'(\mu) v_1^{(2n-1)}(\gamma_*) & u_2^{(2n-2)}(\gamma_*)^* P'(\mu) v_1^{(2n-2)}(\gamma_*) \end{bmatrix}.$$

It easy to see that  $M$  is a indefinite Hermitian matrix, which implies that there exists a nontrivial solution for (7). Suppose that the matrix  $M$  has a spectral decomposition of the form

$$M = U \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} U^*,$$

assume a unit vector  $\begin{bmatrix} \xi \\ \eta \end{bmatrix}$  and set  $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = U \begin{bmatrix} \xi \\ \eta \end{bmatrix}$ . So, the equation (7) turns on

$$\begin{bmatrix} \bar{\xi} & \bar{\eta} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = 0.$$

Finally  $\begin{bmatrix} \xi \\ \eta \end{bmatrix}$  and consequently  $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$  can be obtain form

$$|\xi|^2 \lambda_1 + |\eta|^2 \lambda_2 = 0, \quad \text{and} \quad |\xi|^2 + |\eta|^2 = 1,$$

that straightforward yields

$$\xi = \sqrt{\frac{|\lambda_1|}{|\lambda_1| + |\lambda_2|}}, \quad \text{and} \quad \eta = \sqrt{\frac{|\lambda_2|}{|\lambda_1| + |\lambda_2|}}.$$

Now, return to the above example of normal matrix polynomial. By applying what is discussed for the normal matrix polynomial  $P(\lambda)$  in (2) we have

$$\alpha = -0.6250, \quad \text{and} \quad \beta = -0.7806,$$

also the two vectors  $u(\gamma_*)$  and  $v(\gamma_*)$  in (5) satisfy

$$u_2^*(\gamma_*) P'(\mu) v_1(\gamma_*) = -2.2204 \times 10^{-16},$$

and

$$\|U^*(\gamma_*)U(\gamma_*) - V^*(\gamma_*)V(\gamma_*)\|_2 = 3.3479 \times 10^{-16},$$

Moreover, the matrix polynomial  $Q_{\gamma_*}(\lambda)$  that has  $\mu = 3$  as a multiple eigenvalue can be find as following

$$Q_{\gamma_*}(\lambda) = \begin{bmatrix} 0.7722 & -0.0955 & 0 \\ 0.0527 & 0.6065 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lambda^2 + \begin{bmatrix} -3.2278 & -0.0955 & 0 \\ 0.0527 & -1.3935 & 0 \\ 0 & 0 & 3 \end{bmatrix} \lambda + \begin{bmatrix} 1.7722 & -0.0955 & 0 \\ 0.0527 & -0.3935 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

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